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The SO(3) ⊂ SU(3) problem from a holomorphic induction point of view

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Abstract. A map carrying irreducible representations of SO(3) into an irreducible representation space of SU(3) is given. This map is used to construct orthogonal polynomials of SU(3) in an SO(3) basis. Two procedures are discussed for dealing with multiplicity, one canonical, the other not. It is shown how to construct matrix elements of SU(3) representations in an SO(3) basis.

1. Introduction

Writing the irreducible representations of SU(3) in an SO(3) basis has been a problem of long-standing interest in nuclear physics (Elliott 1958, Harvey 1968; see e.g. Hecht and Zahn 1979 for recent calculations on cluster models); and once it is known how to construct such a basis, it is then possible to compute matrix elements and Wigner coefficients in an SO(3) basis. Of particular mathematical interest is the fact that the SO(3) basis is not multiplicity free. That is, in a given SU(3) irreducible representation, a representation of SO(3) may occur more than once.

This means that operators arising outside SO(3) are necessary to label bases of irreducible representations of SU(3) properly. The state labelling problem has conventionally been dealt with by introducing an operator Ω which commutes with the Lie algebra elements of SO(3) and the Casimir operators of SU(3) (Moshinsky *et al* 1975, Judd *et al* 1974 (where other ways of labelling SO(3) states in SU(3) are also discussed)). Then a state of SU(3) can be written as $|(m)lk\omega\rangle$, where (m) labels an irreducible representation of SU(3), l, k are eigenvalues of the orbital angular momentum and projection, respectively, and ω is the eigenvalue of the operator Ω. Other methods for dealing with the multiplicity problem are given in Moshinsky *et al* (1975) and Judd *et al* (1974).

In this paper we will obtain basis states and compute matrix elements of SU(3) in an SO(3) basis using the methods of holomorphic induction (Klink and Ton-That 1979a and references therein). This will mean that basis elements of SU(3) are realised as polynomials over GL(3, ℂ); and in contrast to the operator Ω, multiplicity will be dealt with using representations of a permutation group (canonical procedure) and with coupled angular momentum (non-canonical procedure).

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To get a feeling for polynomial representations several elementary examples are discussed here which will be of importance later on. The two fundamental (three-dimensional) representations of SU(3), (100) and (110) (better known as the 3 and 3* representations) each contain the three-dimensional $l=1$ representation of SO(3) once. Thus, for these two representations one can write SU(3) states as $|(100)l=1,k\rangle$ and $|(110)l=1,k\rangle$, where $k=0,\pm 1$. It is well known that the (100) and (110) representations of SU(3) can be realised as minors of determinants of GL(3, $\mathbb{C}$) (Klink and Ton-That 1979b). To connect the resulting basis independent polynomial states with Dirac states $|\ \rangle$, it is necessary to embed SO(3) in SU(3) in a definite way. One conventionally uses a spherical basis, which is equivalent to defining SO(3) as that subgroup of SU(3) leaving the form

$$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

invariant. Then SO(2) elements are of the form

$$d=\begin{pmatrix} d & & \\ & 1 & \\ & & d^{-1} \end{pmatrix}, \qquad |d|=1,$$

and

$$T_d|(100)1k\rangle = d^k|(100)1k\rangle, \qquad (T_d\Delta_j^1)(g)=\Delta_j^1(gd)=(gd)_{1j},$$
$$T_d|(110)1k\rangle = d^k|(110)1k\rangle, \qquad (T_d\Delta_{j_1j_2}^{12})(g)=\Delta_{j_1j_2}^{12}(gd), \qquad g\in \mathrm{GL}(3,\mathbb{C}),$$

where

$$\Delta_j^i(g)=g_{ij}, \qquad \Delta_{kl}^{mn}(g)=g_{mk}g_{nl}-g_{ml}g_{nk},$$
$$1\leqslant i,j\leqslant 3, \qquad 1\leqslant m<n\leqslant 3, \qquad 1\leqslant k\leqslant l\leqslant 3.$$

It follows that

$$\left.\begin{aligned} &|(100)1k=1\rangle \to \Delta_1^1(g)=g_{11} \\ &|(100)1k=0\rangle \to \Delta_2^1(g)=g_{12} \\ &|(100)1k=-1\rangle \to \Delta_3^1(g)=g_{13} \end{aligned}\right\}\equiv e_k^{(100)}(g),$$
$$\left.\begin{aligned} &|(110)1k=1\rangle \to \Delta_{12}^{12}(g)=g_{11}g_{12}-g_{12}g_{21} \\ &|(110)1k=0\rangle \to \Delta_{13}^{12}(g)=g_{11}g_{23}-g_{13}g_{21} \\ &|(110)1k=-1\rangle \to \Delta_{23}^{12}(g)=g_{12}g_{23}-g_{13}g_{22} \end{aligned}\right\}\equiv e_k^{(110)}(g). \tag{1}$$

Thus the states of the two three-dimensional representations of SU(3) in an SO(3) basis can be concretely realised as GL(3, $\mathbb{C}$) polynomials.

Once the states are known it is straightforward to compute matrix elements; for as will be shown in § 2, representations of polynomial states are given by right translation: $(T_{g_0}\Delta)(g)=\Delta(gg_0)$, where $\Delta(g)$ is any of the six minors of (1) and $g_0\in$ GL(3, $\mathbb{C}$). From the fact that the minors satisfy the relation $\Delta_j^i(gg_0)=\Sigma_k\,\Delta_k^i(g)\Delta_j^k(g_0)$ and $\Delta_{j_1j_2}^{i_1i_2}(gg_0)=\Sigma_{k_1<k_2}\,\Delta_{k_1k_2}^{i_1i_2}(g)\Delta_{j_1j_2}^{k_1k_2}(g_0)$, it follows that the matrix elements are given

by $\Delta^i_j(g_0)$ for (100) and $\Delta^{i_1 i_2}_{j_1 j_2}(g_0)$ for (110); for example

$$\begin{aligned} \langle(100)1, -1|T_g|(100)1, 1\rangle &= D^{(100)}_{-1+1}(g) = \Delta^2_1(g) = g_{21}, \\ \langle(110)1, 1|T_g|(110)1, -1\rangle &= D^{(110)}_{+1-1}(g) = \Delta^{13}_{23}(g) = g_{12}g_{33} - g_{13}g_{32}. \end{aligned} \tag{2}$$

If the elements $g \in \mathrm{GL}(3, \mathbb{C})$ are restricted to elements of SU(3), then the matrix elements become unitary matrices.

The goal of this paper will be to show how these results can be generalised to arbitrary representations of SU(3). In § 2 arbitrary irreducible representations will be built out of sums of tensor products of the (100) and (110) representations discussed above, with the sums built around certain Wigner coefficients of SO(3). Then in § 3 matrix elements of SU(3) in an SO(3) basis will be computed.

2. SU(3) representations in an SO(3) basis

The goal of this section is to realise the states $|(m)lk\eta\rangle$ as polynomials over $\mathrm{GL}(3, \mathbb{C})$; η is a multiplicity parameter that is to be determined.

We begin by first giving a basis independent definition of polynomial representations of SU(3). Let $(m) \equiv (m_1 \geq m_2 \geq 0)$, m_1, m_2 integers, be a representation of $\mathrm{GL}(3, \mathbb{C})$; then

$$V^{(m)} = \{f: \mathrm{GL}(3, \mathbb{C}) \to \mathbb{C}, f \text{ polynomial}, f(bg) = \pi^{(m)}(b) f(g)\} \tag{3}$$

defines an irreducible polynomial vector space for the representation (m) of $\mathrm{GL}(3, \mathbb{C})$. Here b is an element of the Borel subgroup

$$\mathrm{B} = \left\{ \begin{pmatrix} b_{11} & & 0 \\ & b_{22} & \\ * & & b_{33} \end{pmatrix} \right\},$$

whose representation is given by $\pi^{(m)}(b) = b_{11}^{m_1} b_{22}^{m_2}$. The $\mathrm{GL}(3, \mathbb{C})$ representation is

$$(T_{g_0} f)(g) = f(gg_0), \qquad f \in V^{(m)}, \; g_0 \in \mathrm{GL}(3, \mathbb{C}). \tag{4}$$

A 'differentiation' inner product for $V^{(m)}$ is given by

$$(f, f') = f(\partial/\partial g_{ij}) \overline{f'(\bar{g}_{ij})}|_{g_{ij}} = 0, \tag{5}$$

where the bar denotes complex conjugation. If g_0 in (4) is restricted to the SU(3) subgroup of $\mathrm{GL}(3, \mathbb{C})$, then T_{g_0} remains irreducible in $V^{(m)}$ and becomes unitary with respect to the inner product (5).

We now want to connect the polynomial space $V^{(m)}$ with the fundamental representations discussed in the introduction. To that end consider the m_1-fold tensor product space

$$T^{m_1 m_2} \equiv V^{(100)} \otimes \ldots \otimes V^{(110)}, \tag{6}$$

where the first $m_1 - m_2$ representations are fundamental representations of the form (100) and the remaining m_2 representations are fundamental representations of the form (110). Define the map $\Phi: T^{m_1 m_2} \to V^{(m)}$ by

$$(\Phi F)(g) = F(g, g, \ldots, g), \qquad F \in T^{m_1 m_2}. \tag{7}$$

Then $\Phi T^{m_1 m_2} = V^{(m)}$; this can easily be seen by noting that

$$(\Phi F)(bg) = F(bg, bg, \ldots, bg) = \pi^{(100)}(b) \ldots \pi^{(110)}(b)(\Phi F)(g) = \pi^{(m)}(b)(\Phi F)(g),$$

so ΦF is a polynomial that transforms to the left correctly, and hence is in $V^{(m)}$. For example, if $(m) = (310)$, the 15-dimensional representation of SU(3), we have (uniquely) the three-fold tensor product $(100) \otimes (100) \otimes (110)$. Thus, all representations (m) of SU(3) can be obtained from the tensor product space $T^{m_1 m_2}$ of fundamental representations (100) and (110), using the map Φ.

The goal now is to construct a map $\Lambda_\eta^{m_1 m_2}$ from the irreducible representation space W^l of SO(3) to $T^{m_1 m_2}$. If such a map can be found, then $(\Phi \Lambda_\eta^{m_1 m_2} f)(g), f \in W^l$, will give a polynomial realisation of SU(3) in an SO(3) basis.

To construct such a map, consider another map from W^l to the m_1-fold tensor product space of $l = 1$ representations of SO(3), $\tilde{\Lambda}_\eta^l : W^l \to W^1 \otimes \ldots \otimes W^1$. As shown in the appendix and in Klink (1983), the map is given by

$$\tilde{\Lambda}_\eta^l e_k^l = \sum_{k = k_1 + \ldots + k_{m_1}} \langle 1k_1, \ldots, 1k_{m_1} | lk\eta \rangle e_{k_1}^1 \otimes \ldots \otimes e_{k_{m_1}}^1, \tag{8}$$

where e_k^l is an orthonormal basis element in W^l. $\langle | \rangle$ are SO(3) Wigner coefficients giving the overlap between m_1 $l = 1$ representations with components $k_1, \ldots, k_{m_1}$ and l, k, η, where η as before is a multiplicity index. That is, $\langle | \rangle$ is an orthogonal matrix in $k_1, \ldots, k_{m_1}$ and $lk\eta$. Since $\langle | \rangle$ is orthogonal and $e_{k_1}^1 \otimes \ldots \otimes e_{k_{m_1}}^1$ forms an orthonormal basis in the m_1-fold tensor product space $\tilde{\Lambda}_\eta^l e_k^l$ is also orthonormal.

Assuming now that $\langle | \rangle$ is known (as discussed in the appendix), a Λ map can be defined from W^l to $T^{m_1 m_2}$. The idea is simple; since $V^{(100)}$ and $V^{(110)}$ each contain only the $l = 1$ representation of SO(3), the first $m_1 - m_2$ basis elements $e_{k_i}^1$ of (8) can be replaced by $e_{k_i}^{(100)}$ basis elements of SU(3), and the remaining m_2 elements replaced by $e_{k_i}^{(110)}$. Then

$$\Lambda_\eta^{m_1 m_2} e_k^l = \sum_{k = k_1 + \ldots + k_{m_1}} \langle 1k_1, \ldots, 1k_{m_1} | lk\eta \rangle e_{k_1}^{(100)} \otimes \ldots \otimes e_{k_{m_1}}^{(110)} \tag{9}$$

carries the basis element e_k^l of SO(3) to $T^{m_1 m_2}$ of SU(3).

If $\Lambda_\eta^{n_1 n_2} e_k^l$ is rotated by an element $R \in \mathrm{SO}(3)$, the properties of the Wigner coefficients give

$$\begin{aligned} T_R \Lambda_\eta^{m_1 m_2} e_k^l &= \sum_{k_1 \ldots k_{m_1}} \langle | \rangle (e_{k_1}^{(100)} \otimes \ldots \otimes e_{k_{m_1}}^{(110)})(g_1 R, \ldots, g_{m_1} R) \\ &= \sum_{\substack{k_1 \ldots k_{m_1} \\ k_1' \ldots k_{m_1}'}} \langle | \rangle e_{k_1'}^{(100)} \otimes \ldots \otimes e_{k_{m_1}'}^{(110)} D_{k_1' k_1}^l(R) \ldots D_{k_{m_1}' k_{m_1}}^l(R) \\ &= \sum_{k'} D_{k'k}^l(R) \Lambda_\eta^{m_1 m_2} e_{k'}^l \end{aligned} \tag{10}$$

so that for fixed multiplicity η, $\Lambda_\eta^{m_1 m_2} e_k^l$ transforms irreducibly with respect to SO(3).

The multiplicity index η comes from the m_1-fold tensor product $W^1 \otimes \ldots \otimes W^1$. That is, in such a tensor product space a given irreducible representation l of SO(3) may appear more than once; η labels the different ways in which a given l gets mapped into $T^{m_1 m_2}$. As shown in the appendix, η comes from representations of the group S_{m_1}, the permutation group on m_1 letters.

But only certain representations of S_{m_1} are allowed. This can be seen by composing Φ and $\Lambda_\eta^{m_1m_2}$ so that W^1 is mapped into $V^{(m)}$:

$$\begin{aligned}(\Phi\Lambda_\eta^{m_1m_2}e_k^l)(g)&=\sum_{k_1+\ldots+k_{m_1}=k}\langle|\rangle(\Phi e_{k_1}^{(100)}\otimes\ldots\otimes e_{k_{m_1}}^{(110)})(g),\\ \underset{\eta}{h}{}_{lk}^{(m)}(g)&=\sum_{k_1+\ldots+k_{m_1}}\langle|\rangle e_{k_1}^{(100)}(g)\ldots e_{k_{m_1}}^{(110)}(g),\end{aligned}\tag{11}$$

where $\Phi\Lambda_\eta^{m_1m_2}$ acting on $e_k^l\in W^l$ generates orthogonal (but in general not normalised) basis elements $\underset{\eta}{h}{}_{lk}^{(m)}(g)$ in $V^{(m)}$. That $\underset{\eta}{h}{}_{lk}^{(m)}(g)$ is orthogonal in lk follows from the fact that T_R of (10) intertwines with Φ.

From (11) it is clear that $e_{k_1}^{(100)}(g)\ldots e_{k_{m_1-m_2}}^{(100)}(g)$ is symmetric under the interchange of the m_1-m_2 labels k_i, while similarly the remaining m_2 labels k_i leave the (110) polynomials unchanged. Therefore, only for those representations of S_{m_1} that contain the identity (symmetric) representation of $S_{m_1-m_2}\times S_{m_2}$ will the composition map $\Phi\Lambda_\eta^{m_1m_2}$ be non-zero. Thus, η refers to those irreducible representations of S_{m_1} that contain the identity representations of $S_{m_1-m_2}\times S_{m_2}$. Such a result is similar to that obtained in Klink (1983) for the map sending an irreducible representation of SU(2) into an n-fold tensor product of SU(2) representations.

The simplest class of orthogonal SO(3) polynomials comes from those representations of SU(3) for which there is no multiplicity. For SU(3) representations of the form $(m, 0, 0)$ or $(m, m, 0)$, the SO(3) content is given by $l=m, m-2, m-4\ldots.$ From equation (11) we get

$$h_{lk}^{(m00)}(g)=\sum_{k_1+\ldots+k_m=k}\langle|\rangle e_{k_1}^{(100)}(g)\ldots e_{k_m}^{(100)}(g),\tag{12}$$

where the SO(3) Wigner coefficients (given in equation (A5)) are symmetric under any interchange of $k_1\ldots k_m$ (they transform as the identity representation of S_m). For the representation $(mm0)$ it is merely necessary to replace the (100) representations by (110) representations in equation (12).

For example, $(m)=(200)$ contains $l=2, 0$ and the Wigner coefficients of interest are $\langle 1k_11k_2|lk\rangle$ with $\langle|\rangle$ symmetric under the interchange of k_1 and k_2. This gives

$$\begin{aligned}
&h_{2,2}^{(200)}=g_{11}^2, && h_{2,2}^{(220)}=[\Delta_{12}^{12}(g)]^2,\\
&h_{2,1}^{(200)}=g_{11}g_{12}, && h_{2,1}^{(220)}=\Delta_{12}^{12}(g)\Delta_{13}^{12}(g),\\
&h_{2,0}^{(200)}=g_{11}g_{13}+g_{12}^2, && h_{2,0}^{(220)}=\Delta_{12}^{12}(g)\Delta_{23}^{12}(g)+[\Delta_{13}^{12}(g)]^2,\\
&h_{2,-1}^{(200)}=g_{13}g_{12}, && h_{2,-1}^{(220)}=\Delta_{13}^{12}(g)\Delta_{23}^{12}(g),\\
&h_{2,-2}^{(200)}=g_{13}^2, && h_{2,-2}^{(220)}=[\Delta_{23}^{12}(g)]^2,\\
&h_{0,0}^{(200)}=2g_{11}g_{13}-g_{12}^2, && h_{0,0}^{(220)}=2\Delta_{12}^{12}(g)\Delta_{23}^{12}(g)-[\Delta_{13}^{12}(g)]^2,
\end{aligned}\tag{13}$$

where $\Delta_{ij}^{12}(g)$ are the minors given in (1). The orthogonality of these basis elements is easily checked using the differentiation inner product, (5); similarly the factors which normalise the polynomials (13) are obtained using (5).

Using (12), it is possible to construct orthogonal polynomials $\underset{\eta}{h}{}_{lk}^{(m)}(g)$ for any representation (m) of SU(3). One simply obtains a representation $(m)=(m_1, m_2, 0)$ from the tensor product $(m_1-m_2, 0, 0)\otimes(m_2, m_2, 0)$ via a map Φ from

$V^{(m_1-m_2,0,0)} \otimes V^{(m_2,m_2,0)}$ to $V^{(m_1,m_2,0)}$ defined by

$$(\Phi F)(g)=F(g,g), \qquad F\in V^{(m_1-m_2,0,0)}\otimes V^{(m_2,m_2,0)}.$$

That $\Phi F\in V^{(m_1,m_2,0)}$ requires showing that ΦF satisfies the definition of $V^{(m)}$ given in (3); the proof follows that given in equation (7).

However, $(\Phi e_{l_1k_1}^{(m_1-m_2,0,0)}\otimes e_{l_2k_2}^{(m_2,m_2,0)})(g)$ does not transform as an irreducible representation l of SO(3). But by coupling l_1 to l_2 we get

$$\underset{(l_1,l_2)}{h^{(m)}_{l,k}}(g)=\sum_{k_1,k_2}\langle l_1k_1l_2k_2|lk\rangle\Phi(e_{l_1k_1}^{(m_1-m_2,0,0)}\otimes e_{l_2k_2}^{(m_2,m_2,0)})(g), \tag{14}$$

where $\langle l_1k_1l_2k_2|lk\rangle$ is an SO(3) Wigner coefficient; notice that $e_{l_1k_1}^{(m_1-m_2,0,0)}$ and $e_{l_2k_2}^{(m_2,m_2,0)}$ must be the properly normalised polynomials. For example, in (13), $h_{2,-2}^{(200)}=g_{13}^2$ becomes $e_{2,-2}^{(200)}=(1/\sqrt{2})g_{13}^2$.

It may perhaps seem strange that η in (11) refers to representations of S_{m_1} while the multiplicity in (14) is given by (l_1,l_2). For representations (m) for which the multiplicity of a given representation l of SO(3) is one, the two polynomials agree up to a normalisation factor. However, (14) is computationally much simpler because only relatively simple Wigner coefficients are required. In contrast, the SO(3) Wigner coefficients needed in (11) are more difficult to compute, because (. their required transformation properties under S_{m_1}. When a given representation l occurs more than once in (m), the polynomials (11) and (14) do not agree. While the labels (l_1,l_2) in (14) are sufficient to resolve the multiplicity, the polynomials (for a fixed l,k) are linearly independent, but not orthogonal. In contrast the polynomials of (11) are orthogonal in η because the Wigner coefficients transform irreducibly with respect to S_{m_1}. We call the labelling in (11) canonical and in (14) non-canonical. Thus, the canonical polynomials of (11) are orthogonal in l, k, η but more difficult to obtain than the non-canonical polynomials (14).

As an example of these considerations, we study the representation (420) which contains $l=4,3,2,2,0$; note that $l=2$ has multiplicity 2. Then $m_1=m_2=2$, so that the polynomials $e_{l_1k_1}^{(200)}$ and $e_{l_2k_2}^{(220)}$ are needed to compute the non-canonical polynomials $\underset{(l_1l_2)}{h^{(420)}_{lk}}(g)$. For simplicity only polynomials with $k=0$ will be given. The Wigner coefficients needed in (14) are:

$k=0$	$(k_1\ k_2)$ 2 −2	−2 2	1 −1	−1 1	0 0	N
$l=4$	1	1	4	4	6	$1/\sqrt{70}$
3	1	−1	2	−2	0	$1/\sqrt{10}$
2	2	2	1	1	−2	$1/\sqrt{14}$
1	2	−2	−1	1	0	$1/\sqrt{10}$
0	1	1	−1	−1	1	$1/\sqrt{5}$

where N is a factor needed to normalise the Wigner coefficients. Then, for example,

$$\begin{aligned}\underset{(2,2)}{h^{(420)}_{0,0}}&=e_{2,2}^{(200)}(g)e_{2,-2}^{(220)}(g)+e_{2,-2}^{(200)}(g)e_{2,2}^{(220)}(g)-e_{2,1}^{(200)}(g)e_{2,-1}^{(220)}(g)\\&\quad-e_{2,-1}^{(200)}(g)e_{2,1}^{(220)}(g)+e_{2,0}^{(200)}(g)e_{2,0}^{(220)}(g)\\&=(2g_{11}g_{13}-g_{12}^2)(2\Delta_{12}^{12}(g)\Delta_{23}^{12}(g)-[\Delta_{13}^{12}(g)]^2).\end{aligned} \tag{15}$$

It is also possible to reach $l, k = 0, 0$ by $l_1 = l_2 = 0$. Then $h^{(420)}_{\substack{0,0\\(0,0)}} = e^{(420)}_{0,0}(g)e^{(220)}_{0,0}(g)$ and when the polynomials (13) are used, the result agrees with (15).

That $h^{(420)}_{\substack{0,0\\(0,0)}}$ agrees with $h^{(420)}_{\substack{0,0\\(2,2)}}$ comes about because there is only one way that $l = 0$ can sit in (420). But when there is multiplicity things become more complicated. For the $l = 2$ representations in (420) there are three ways to couple l_1 to l_2 to obtain $l = 2$, namely $2 \otimes 2$, $0 \otimes 2$ and $2 \otimes 0$. Each of these possibilities will result in polynomials, namely $h^{(420)}_{\substack{2,k\\(2,2)}}$, $h^{(420)}_{\substack{2,k\\(0,2)}}$, and $h^{(420)}_{\substack{2,k\\(2,0)}}$, but only two are linearly independent, and none are orthogonal to each other, although they are of course orthogonal to polynomials with different l values. Orthogonal polynomials may be generated by using a Gram–Schmidt process, but there is no unique way of choosing the $(l_1 l_2)$ labels.

To resolve the multiplicity canonically, Wigner coefficients of the form $\langle 1k_1 1k_2 1k_3 1k_4 | lk\eta\rangle$ are needed, where η refers to representations of S_4 and to basis labels in these representations which contain the identity representation of $S_2 \times S_2$. These Wigner coefficients are more difficult to obtain than those needed for the non-canonical bases, but they do generate polynomials that are orthogonal in η.

3. Matrix elements of SU(3) in an SO(3) basis

The polynomials $h^{(m)}_{\substack{lk\\ \eta}}$ are orthogonal, but not normalised. The normalised polynomials will be designated by $e^{(m)}_{lk\eta}(g)$, so that

$$e^{(m)}_{lk\eta}(g) = h^{(m)}_{\substack{lk\\ \eta}}(g)/\|h^{(m)}_{\substack{lk\\ \eta}}\|, \tag{16}$$

where the norm $\| \; \|$ is given in (5). As stated in § 2 the action of $g_0 \in SU(3)$ on the orthonormal polynomial (16) is given by

$$(T_{g_0} e^{(m)}_{lk\eta})(g) = e^{(m)}_{lk\eta}(gg_0).$$

The matrix elements in an SO(3) basis are then

$$D^{(m)}_{lk\eta l'k'\eta'}(g_0) \equiv (e^{(m)}_{lk\eta}, T_{g_0} e^{(m)}_{l'k'\eta'}). \tag{17}$$

Although (17) is the usual definition of a matrix element, it does not provide the most convenient means by which to compute the matrix element.

To find a convenient way of computing the matrix element (17), we return to the tensor product space $T^{m_1 m_2}$ (equation (6)) and note (using Dirac notation) that

$$T_{g_0}|(100)1k_1\rangle \ldots |(110)1k_{m_1}\rangle = \sum_{k'_1 \ldots k'_{m_1}} D^{(100)}_{k'_1 k_1}(g_0) \ldots D^{(110)}_{k'_{m_1} k_{m_1}}(g_0) \times |(100)1k'_1\rangle \ldots |(110)1k'_{m_1}\rangle,$$

where $D^{(100)}_{k'_1 k_1}(g_0)$ etc are the matrix elements of (100) given in the introduction, (2). Now

$$|(m)lk\eta\rangle = \sum_{k_1 \ldots k_{m_1}} \langle (100)k_1 \ldots (110)k_{m_1} | (m)lk\eta\rangle |(100)1k_1\rangle \ldots |(110)1k_{m_1}\rangle,$$

where $\langle | \rangle$ is a Wigner coefficient for coupling $(100) \otimes \ldots \otimes (110)$ to obtain (m). Right translating by g_0 and taking the inner product gives

$$\begin{aligned} D^{(m)}_{l'k'\eta' lk\eta}(g_0) &\equiv \langle (m) l'k'\eta' | T_{g_0} | (m) lk\eta \rangle \\ &= \sum_{\substack{k_1 \ldots k_{m_1} \\ k'_1 \ldots k'_{m_1}}} \langle (m) l'k'\eta' | (100) k'_1 \ldots (110) k'_{m_1} \rangle \\ &\quad \times D^{(100)}_{k'_1 k_1}(g_0) \ldots D^{(110)}_{k'_{m_1} k_{m_1}}(g_0) \langle (100) k_1 \ldots (110) k_{m_1} | (m) lk\eta \rangle, \end{aligned} \tag{18}$$

so that the matrix element $D^{(m)}_{l'k'\eta' lk\eta}(g_0)$ is given by Wigner coefficients, which must be calculated, and matrix elements of the fundamental representations, which were obtained in the introduction. From (18) one sees that $D^{(m)}_{l'k'\eta' lk\eta}(g_0)$ is a polynomial in $g_0 \in \mathrm{SU}(3)$.

The Wigner coefficients needed in (18) can be calculated by defining a map $\alpha^{\bar{m}}_{\bar{\eta}}$ from $V^{(\bar{m})}$ to $T^{m_1 m_2}$ such that

$$\alpha^{\bar{m}}_{\bar{\eta}} e^{\bar{m}}_{\bar{k}} = \sum_{k_1 \ldots k_{m_1}} \langle (\bar{m}) \bar{k} \bar{\eta} | (100) k_1 \ldots (110) k_{m_1} \rangle e^{(100)}_{k_1} \otimes \ldots \otimes e^{(110)}_{k_{m_1}} \tag{19}$$

with the inverse

$$e^{(100)}_{k_1} \otimes \ldots \otimes e^{(110)}_{k_{m_1}} = \sum_{(\bar{m}) \bar{k} \bar{\eta}} \langle (\bar{m}) \bar{k} \bar{\eta} | (100) k_1 \ldots (110) k_{m_1} \rangle \alpha^{(\bar{m})}_{\bar{\eta}} e^{(\bar{m})}_{\bar{k}}; \tag{20}$$

here $(\bar{m})$ denotes those representations of SU(3) that appear in $T^{m_1 m_2}$, $\bar{\eta}$ is a degeneracy parameter, and $\bar{k}$ a basis label in the space $(\bar{m})$. Applying the operator Φ defined in (7) to both sides of (20) gives

$$\Phi(e^{(100)}_{k_1} \otimes \ldots \otimes e^{(110)}_{k_{m_1}}) = \sum_{\bar{k}} \langle | \rangle e^{(m)}_{\bar{k}}; \tag{21}$$

that is, Φ has the property (as shown in equation (7) ff) of projecting out just the highest weight representation (m). If the basis label $\bar{k}$ is now chosen to be the SO(3) basis labels $lk\eta$, the orthonormality properties of $e^{(m)}_{lk\eta}$ give

$$K^{(m)\,(100) \ldots (110)}_{lk\eta\,k_1 \quad\; k_{m_1}} = (e^{(m)}_{lk\eta}, \Phi(e^{(100)}_{k_1} \otimes \ldots \otimes e^{(110)}_{k_{m_1}})), \tag{22}$$

where K^{-}_{-} is a Clebsch–Gordan coefficient (an unnormalised Wigner coefficient)—unnormalised because in general Φ does not preserve norms. In fact, since Φ does not preserve norms, to obtain the desired Wigner coefficients, it suffices to replace $e^{(m)}_{lk\eta}$ in (22) by $\Phi \Lambda^{m_1 m_2}_{\eta} e^{l}_{k}$, compute K^{-}_{-}, and then normalise the K^{-}_{-} coefficients. Thus (22) provides an explicit procedure for computing the Wigner coefficients (21) and hence computing matrix elements (18), for the orthogonal polynomials $\Phi \Lambda^{m_1 m_2}_{\eta} e^{l}_{k}$ are given by (11), the fundamental representations are given in the introduction, and the inner product is the 'differentiation' inner product, (5).

As an example of these considerations, we compute some matrix elements of (310) in an SO(3) basis. We need Clebsch–Gordan coefficients of the form

$$K^{(310)\,(100)\,(100)\,(110)}_{lk\eta\;\; k_1 \;\; k_2 \;\; k_3} = (h^{(310)}_{\substack{lk \\ \eta}}, \Phi(e^{(100)}_{k_1} \otimes e^{(100)}_{k_2} \otimes {}^{(110)}_{k_3})).$$

For example, for $l, k = 3, 0$, the Clebsch–Gordan coefficient becomes

$$
\begin{array}{lccccccc}
k_1k_2k_3 = & 000 & 10-1 & 01-1 & 0-11 & -101 & 1-10 & -110 \\
K^{(310)}_{3,0}\,{}^{(100)}_{k_1}\,{}^{(100)}_{k_2}\,{}^{(110)}_{k_3} & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\
\langle (310)3, 0 | k_1k_2k_3 \rangle = & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}}
\end{array}
$$

and this suffices to give $D^{(310)}_{30^{30}_{2}30^{30}_{2}}(g)$ using (18). Note that no η label is needed here since $l = 3$ has multiplicity one in (310).

4. Conclusion

We have shown how to construct orthogonal GL(3, ℂ) polynomials for SU(3) in an SO(3) basis, using two different procedures. The first, called the canonical procedure, makes use of maps $\Phi\Lambda^{m_1m_2}_{\eta}$ (equation (11)) carrying basis elements e^{l}_{k} of SO(3) into the representation space $(m) = (m_1m_20)$ of SU(3). The multiplicity label η comes from representations of an underlying permutation group S_{m_1}, and guarantees that if the multiplicity of l in (m) is greater than one, the polynomials will be orthogonal in η.

The non-canonical procedure makes use of the fact that the multiplicity of l in $(m00)$ or $(mm0)$ is always one or zero; also the SO(3) Wigner coefficients needed for $\Phi\Lambda^{(m00)}$ are relatively easy to obtain. Then the polynomials for an arbitrary representation (m) of SU(3) are obtained with the help of simple Wigner coefficients of SO(3). If l occurs more than once in (m), the multiplicity is labelled by the angular momenta l_1 of $(m_1 - m_2, 0, 0)$ and l_2 of $(m_2, m_2, 0)$, but the resulting polynomials (equation (14)) are not orthogonal in the multiplicity variables (l_1, l_2). This procedure is called non-canonical because, though the polynomials are linearly independent in (l_1, l_2) and so via a Gram–Schmidt process can be made orthogonal, there is no unique or canonical procedure for carrying out the orthogonalisation process. While this is a disadvantage in comparison with the canonical procedure, the non-canonical procedure has the advantage that the required SO(3) Wigner coefficients are much easier to calculate. In particular, when l occurs in (m) only once, the two procedures must agree, and then it is easier to exhibit the actual GL(3, ℂ) polynomials using the non-canonical procedure.

Once the Wigner coefficients are known, the orthogonal GL(3, ℂ) polynomials $h^{(m)}_{\substack{lk\\ \eta}}$ (see (11)) can be used to compute matrix elements of SU(3) in an SO(3) basis (equation (18)) and also other coefficients of interest in nuclear physics. For example, if $e^{(m)}_{[k]}(g)$ are Gelfand–Cetlin basis elements for the representation (m)—that is, basis elements defined with respect to the subgroup chain SU(3) ⊃ U(2) ⊃ U(1)—then $(e^{(m)}_{[k]}, e^{(m)}_{lk\eta})$ provide the transformation coefficients between the two types of bases. Here $(\cdot, \cdot)$ is again the differentiation inner product (5), and the Gelfand–Cetlin basis realised as polynomials over GL(3, ℂ) is given in Nagel and Moshinsky (1965) and Klink and Ton-That (1982).

Thus, writing an SU(3) representation in an SO(3) basis involves computing the Wigner coefficients for the m_1-fold tensor product $1 \otimes \ldots \otimes 1$. Problems here include finding a simple way in which to obtain these coefficients from a computer. Ideally one would like to choose a representation (m) of SU(3), calculate the l content of this representation and then compute the desired Wigner coefficients. As discussed in the appendix and in Klink (1983), this also probably involves differentiating polynomials, and the goal is to find the most efficient way for doing this.

Also, a closer analysis of the multiplicity label η is required. For a given m_1-fold tensor product of $l=1$ representations, it does not seem possible to label all the representations of SO(3) using representations of the underlying S_{m_1} group. This aspect of the multiplicity problem is closely related to the notion of plethysm (Wybourne 1970). However, not all representations of l occurring in the m_1-fold tensor product are needed; as discussed in § 2, only those representations for which $S_{m_1-m_2}\times S_{m_2}$ carries the identity representation give a non-zero $\Lambda_\eta^{(m)}$ map. How these two aspects of the multiplicity problem mix together needs to be more carefully investigated. For example, in the six-fold tensor product $1\otimes\ldots\otimes 1$, there is a 42 representation of S_6 which serves to break the multiplicity of $l=1$ representations; but (420) of SU(3) also contains $l=2$ twice, so 42 does not uniquely resolve the multiplicity. On the other hand, $l=2$ representations injected into (420) have a definite symmetry with respect to S_4 representations. It is not clear whether the multiplicity can always be resolved using such a 'recursive' definition of symmetry type.

Appendix. Wigner coefficients for n-fold $l=1$ representations

As shown in § 2, an essential part of the $\Lambda_\eta^{m_1m_2}$ map involves the Wigner coefficients for converting m_1 $l=1$ representations to a direct sum of representations of SO(3), labelled by a multiplicity parameter η. In this appendix we first wish to see how η can be obtained, and then further, see how the Wigner coefficients can be calculated.

Finding the multiplicity in an n-fold tensor product of (100) representations of SU(3) is straightforward; it is simply necessary to find the dimension of the corresponding S_n representation. For example, in $(100)\otimes(100)\otimes(100)$, the representation (210) of SU(3) occurs twice because the dimension of the corresponding 21 representation of S_3 is 2. But it is also straightforward to compute the multiplicity of l in a given representation (m) of SU(3) (Moshinsky *et al* 1975, Judd *et al* 1974). Then the multiplicity of a given l occurring in the n-fold tensor product of $l=1$ representations is the product of the dimension of the S_n representation times the multiplicity of l in the corresponding SU(3) representation, summed over all SU(3) representations. Stated in this way, the multiplicity is not easily expressed in closed form; a closed form expression using other methods (Mikhailov 1977, Rashid 1977) is given by

$$P_{jn}^{l}=\sum_k(-1)^k\binom{n}{k}\binom{2n-3k-j-2}{n-2}, \tag{A1}$$

where j is a representation of SU(2), and P_{jn}^{l} is the multiplicity of j occurring in the n-fold product $1\otimes 1\otimes\ldots\otimes 1$. However, our interest here is not in finding a general expression for the multiplicity, but rather in finding a computational procedure for obtaining the Wigner coefficients. By using the fact that $l=1$ is irreducible in (100) of SU(3), we are able to associate with each l occurring in the n-fold tensor product an irreducible representation of S_n which can be used for computing the Wigner coefficients.

The Wigner coefficients needed (in (8)) are $\langle 1k_1\ldots 1k_n|lk\eta\rangle$. These coefficients are required to have special transformation properties under S_n; to see what these transformation properties are, define

$$T_p(e^1_{k_1}\otimes\ldots\otimes e^1_{k_n})\equiv e^1_{p(k_1)}\otimes\ldots\otimes e^1_{p(k_n)}, \tag{A2}$$

where $p\in S_n$, and $p(k_i)$ means the permutation of the ith entry. We also demand, in equation (8), that

$$T_p\tilde{\Lambda}^l_\eta e^l_k=\sum_{\eta'} d_{\eta'\eta}(p)\tilde{\Lambda}^l_{\eta'}e^l_k, \tag{A3}$$

where $d_{\eta'\eta}(p)$ is a matrix element of S_n. With these requirements it is clear from (8) that $\langle 1k_1\ldots 1k_n|lk\eta\rangle$ will transform in its left entry as the inverse of (A2) and in the right entry as (A3). The transformation properties under T_p are preserved for $\Lambda^{m_1m_2}_\eta$ and $\Phi\Lambda^{m_1m_2}_\eta$, so that

$$T_p\Phi\Lambda^{m_1m_2}_\eta e^l_k=\sum_{\eta'} d_{\eta'\eta}(p)\Phi\Lambda^{m_1m_2}_{\eta'}e^l_k, \tag{A4}$$

and from (A4) it follows, using the orthogonality properties of the matrix elements of S_{m_1}, that $h^{(m)}_{\substack{lk\\ \eta}}$ (equation (11)) is orthogonal in η.

We now wish to compute Wigner coefficients with the desired transformation properties. In Klink (1983) it was shown how one can obtain Wigner coefficients for an n-fold tensor product $j_1\otimes\ldots\otimes j_n$ of SU(2); here we wish to apply this method to $1\otimes 1\otimes\ldots\otimes 1$. Now each j has corresponding to it a Gelfand label $m=2j$, so the tensor product of interest is actually $m_1\otimes\ldots\otimes m_n$. A map defined analogously to (7) sends elements of a space $T^r_{(10)}\equiv V^{(10)}\otimes\ldots\otimes V^{(10)}$, $r=\sum_{i=1}^n m_i$, to the desired tensor product space. The multiplicity of a given representation j in $j_1\otimes\ldots\otimes j_n$ is closely connected with the dimension of a representation of the permutation group S_r. In fact, Klink (1983) shows that the multiplicity of j in $j_1\otimes\ldots\otimes j_n$ is given by the number of times the identity representation of the subgroup $S_{m_1}\times\ldots\times S_{m_n}$ occurs in the corresponding representation of S_r. In the case of interest in this paper, where $l=1\to m=2$, the multiplicity is given by the number of times the identity representation of $S_2\times\ldots\times S_2$ occurs in $S_r=S_{2n}$. However, this is the multiplicity independent of any further symmetry. Unlike $j_1\otimes\ldots\otimes j_n$, where there is in general no permutational symmetry of the representations, in the case of $1\otimes\ldots\otimes 1$, all the representations are the same, and a permutational symmetry provides at least part of the multiplicity label. This means that, unlike the general case where a Gelfand pattern specifies a matrix element of S_r which can be used to compute Wigner coefficients, we must introduce a subgroup scheme in S_r that both restricts the representations of $S_2\times\ldots\times S_2$ in S_r to be the identity representation, and also allows for permutation symmetry of the n $l=1$ representations. Such a group is precisely the semidirect product group $G\equiv(S_2\times\ldots\times S_2)\circledS S_n$, which is a subgroup of $S_r=S_{2n}$. The representations of G are easily obtained via the Mackey induced representation theory, but whether they provide the simplest and computationally most effective way to obtain the projection operators needed for the Wigner coefficients still remains to be seen. The conclusion to be drawn here is that the methods of Klink (1983) for computing Wigner coefficients can be used to compute those Wigner coefficients of interest in this paper, but whether such a method is the simplest one is not known.

However, when the Wigner coefficients are $\langle 1k_1\ldots 1k_n|lk\rangle$ with $\langle|\rangle$ invariant under any interchange of $k_1\ldots k_n$ (so that η is the identity (symmetric) representation of S_n), then the coefficients are much easier to obtain; such Wigner coefficients are needed for $e^{(n00)}_{lk}$ and $e^{(nn0)}_{lk}$ (see equation (12)). To begin, the only symmetric representation of S_n in the n-fold tensor product $(100)\otimes\ldots\otimes(100)$ of SU(3) is $(n00)$; hence, the only symmetric representations l of SO(3) in the n-fold tensor product $1\otimes\ldots\otimes 1$ are given by the SO(3) content of $(n00)$, which is $l=n, n-2, n-4, \ldots$. Therefore, if $\langle 1k_1\ldots 1k_n|lk\rangle$ is invariant under the interchange of $k_1\ldots k_n$, only SO(3) representations of the form $n, n-2, \ldots$ can occur.

But the Wigner coefficients $\langle 1k_1\ldots 1k_n|nk\rangle$ are easily obtained using SO(3) lowering operators, starting with

$$|n,n\rangle=|1,1\rangle\ldots|1,1\rangle. \tag{A5}$$

Applying the lowering operator to both sides of (A5) gives

$$|n,n-1\rangle=N\{|1,0\rangle|1,1\rangle\ldots|1,1\rangle+\ldots+|1,1\rangle\ldots|1,0\rangle\}\equiv N|\{11\ldots10\}\rangle,$$

where N is a normalisation factor, and $\{11\ldots10\}$ stands for the set of variables $k_1, k_2, \ldots, k_n$ such that $k_1+k_2+\ldots+k_n=n-1$. Lowering again gives

$$|n,n-2\rangle=N'[|\{11\ldots100\}\rangle+|\{11\ldots1-1\}\rangle],$$

and the state $|n-2,n-2\rangle$ must be orthogonal to $|n,n-2\rangle$. This fixes the Wigner coefficients to be

(A6)

$k=n-2$	$(k_1\ldots k_n)$ $\{11\ldots100\}^{(n-1)(n-2)}$	$\{11\ldots1-1\}^{(n)}$	N
$l=n$	2	1	$[4n^2-11n+8]^{-1/2}$
$l=n-2$	1	$-(n-1)$	$[(n-1)(n^2-2)]^{-1/2}$

where the superscript on { } gives the number of different ways $k_1+\ldots+k_n=n-2$, first with $n-2$ '1's' and two '0's', and then $n-1$ '1's' and one '-1'. N is the normalisation coefficient for the Wigner coefficients.

Proceeding further, the states $|n,n-2\rangle$ and $|n-2,n-2\rangle$ are lowered to $|n,n-4\rangle$ and $|n-2,n-4\rangle$, where a new state $|n-4,n-4\rangle$ appears. Requiring that this state be orthogonal to the previous two then fixes the $|n-4,n-4\rangle$ Wigner coefficients. This procedure can be continued until all of the Wigner coefficients in the chain $l=n$, $n-2$, $n-4$ are computed.

As an example consider $n=4$; the Wigner coefficients for $k=2$ and $k=0$ are

(A7)

$k=2$	$(k_1k_2k_3k_4)$ $\{1100\}^{(6)}$	$\{111-1\}^{(4)}$	N	
$l=4$	2	1	$1/\sqrt{28}$	
$l=2$	1	-3	$1/\sqrt{42}$	
$l=0$	$(k_1k_2k_3k_4)$ $\{0000\}^{(1)}$	$\{11-1-1\}^{(6)}$	$\{100-1\}^{(12)}$	N
$l=4$	4	1	2	$1/\sqrt{70}$
$l=2$	12	-4	-1	$1/\sqrt{252}$
$l=0$	3	2	-1	$1/\sqrt{45}$

where as before $\{111-1\}^{(4)}$ means the four states $|1\rangle|1\rangle|1\rangle|-1\rangle\ldots|-1\rangle|1\rangle|1\rangle|1\rangle$ etc. These coefficients can be used to obtain the polynomials $e_{lk}^{(400)}$ and $e_{lk}^{(440)}$, using (12).

References

Elliott J P 1958 *Proc. R. Soc.* A **245** 128, 562
Harvey M 1968 *Advances in Nuclear Physics* vol 1, ed M Baranger and E Vogt (New York: Plenum)
Hecht K T and Zahn W 1979 *Nucl. Phys.* A **318** 1, A **313** 77
Judd B R, Miller Jr W, Patera J and Winternitz P 1974 *J. Math. Phys.* **15** 1787
Klink W H 1983 *J. Phys. A: Math. Gen.* **16** 1845–54
Klink W H and Ton-That T 1979a *Ann. Inst. Henri Poincaré* A **31** 77
—— 1979b *CR Acad. Sci. Ser. B* **289** 115
—— 1982 *Ann. Inst. Henri Poincaré* A **36** 225
Mikhailov V V 1977 *J. Phys. A: Math. Gen.* **10** 147
Moshinsky M, Patera J, Sharp R T and Winternitz P 1975 *Ann. Phys., NY* **95** 139
Nagel J G and Moshinsky M 1965 *J. Math. Phys.* **6** 682
Rashid M A 1977 *J. Phys. A: Math. Gen.* **10** L135
Wybourne B G 1970 *Symmetry Principles and Atomic Spectroscopy* (New York: Wiley Interscience) ch 6